9.3: Rate of Change

Example: Ron and Harry go for a walk. Here is the total distance they have traveled at various times:

t (hours)	0	0.5	1.0	1.5	2.0
$D(t)$ (miles)	0	8	15	22	50

The ***rate of change*** of distance is the ratio (fraction) of change in distance over time. Briefly,

$$Rate = \frac{\Delta Dist}{\Delta Time}$$

We use the word "speed" when talking about "rate of change of distance"

Q: What is the (overall) average speed from t = 0 to t = 2 hours?

$$\text{SPEED} = \frac{\Delta \text{DIST}}{\Delta \text{TIME}} = \frac{D(2) - D(0)}{2 - 0}$$

$$= \frac{50 - 0}{2 - 0}$$

$$= \frac{50 \text{ miles}}{2 \text{ hours}}$$

$$= \boxed{25 \text{ mph}}$$

Q: What is the average speed from t = 1.5 to t = 2 hrs?

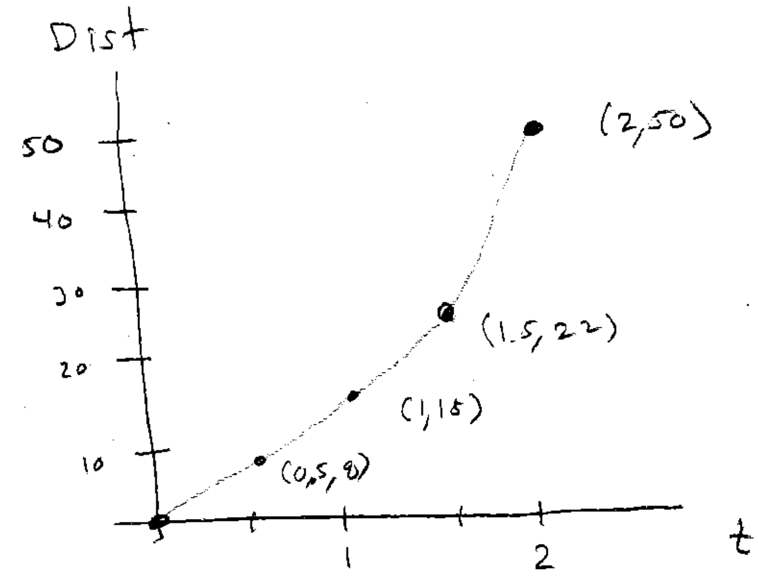
$$\text{SPEED} = \frac{D(2) - D(1.5)}{2 - 1.5}$$

$$= \frac{50 - 22}{2 - 1.5}$$

$$= \frac{28 \text{ miles}}{0.5 \text{ hours}} = \boxed{56 \text{ mph}}$$

Q: What is the speedometer speed at t = 2 hrs?

"Probably" closer to 56 mph than 25 mph.



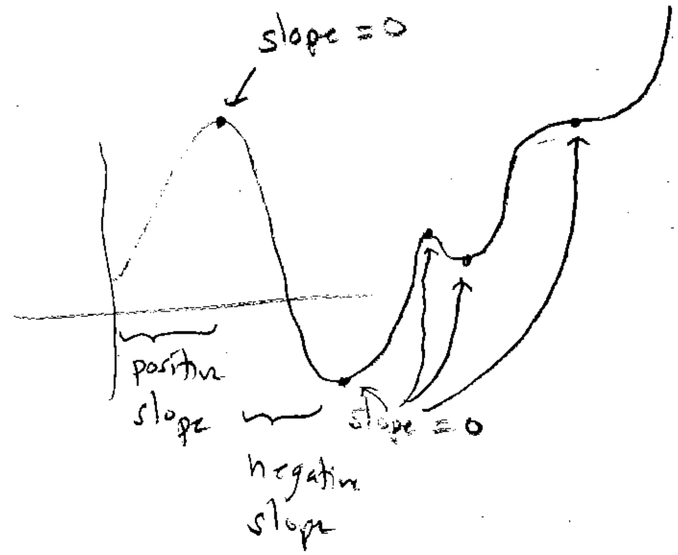
A: We don't have enough information to say for sure, we can only estimate.

We call this the ***instantaneous rate of change of distance at t = 2.***

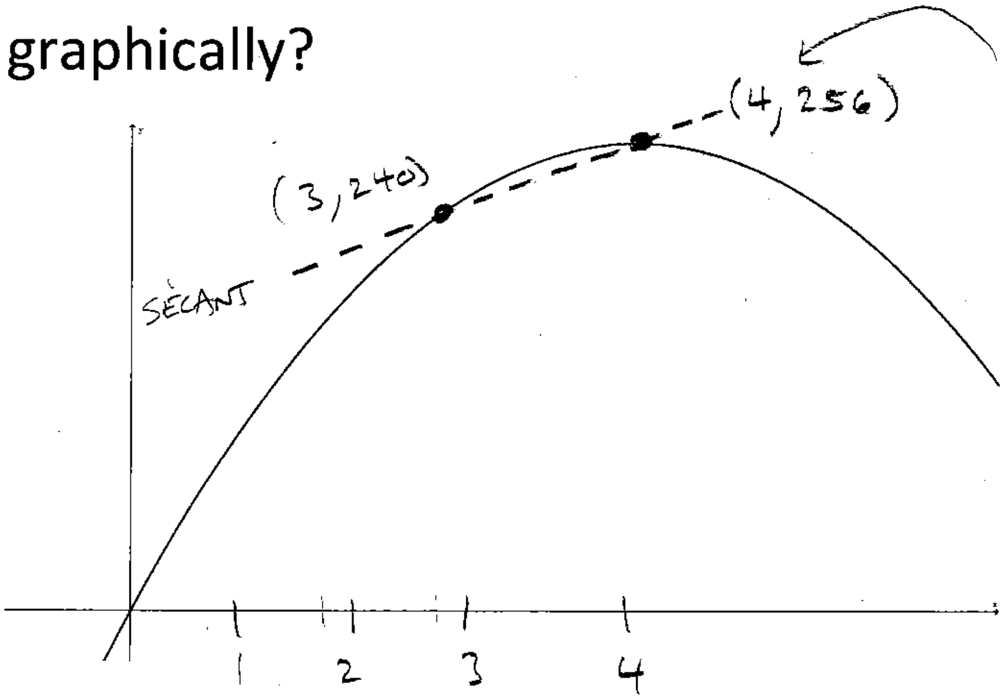
(or just the rate of change **at** t = 2).

Math 112 is all about instantaneous rates of change and the many things we know about them. We will:

1. Develop tools to quickly find rates at a point.
2. Use these tools to go between our business functions.
3. Use these tools to analyze our business functions (max/min, increasing/decreasing, and more).
4. Learn the language of rates and calculus which you need in business and economics.



How would you do the last question graphically?



NOTE/Review:

A *secant* line goes through a graph at two points. A *tangent* line just touches a graph at one point with the same slope as the graph at that point.

slope of secant = average rate

$$= \frac{f(b)-f(a)}{b-a}$$

slope of tangent = instantaneous rate

$$= f'(a)$$

Now assume we have an algebraic rule instead of a table:

Example: Assume Tommy is in a train. His distance from the starting line (in feet) after t seconds is given by

$$D(t) = 128t - 16t^2$$

Q: What is the average speed from t = 3 to t = 4 seconds?

$$\text{RATE} = \frac{\Delta\text{DIST}}{\Delta\text{TIME}}$$

$$= \frac{D(4) - D(3)}{4 - 3}$$

$$= \frac{[128(4) - 16(4)^2] - [128(3) - 16(3)^2]}{4 - 3}$$

$$= \frac{256 - 240}{4 - 3} = \frac{16 \text{ feet}}{1 \text{ sec}} = \boxed{16 \ \frac{\text{ft}}{\text{sec}}}$$

Q: What is the instantaneous speed at t = 3 seconds?

We give this the notation:

$D'(3)$ = "instantaneous speed at t=3"

We don't know the tools to find this exactly yet, but, we can approximate:

Idea: Let's find the average speed from t = 3 to t = 3.01 seconds and use that as an approximation.

$D'(3) \approx$ AVERAGE SPEED FROM $t=3$ to $t=3.01$

$$= \frac{D(3.01) - D(3)}{3.01 - 3}$$

$$= \frac{[128(3.01) - 16(3.01)^2] - [128(3) - 16(3)^2]}{3.01 - 3}$$

$$= \frac{240.3184 - 240}{3.01 - 3}$$

$$= \frac{0.3184 \text{ ft}}{0.01 \text{ sec}}$$

$$= \boxed{31.84 \text{ ft/sec}}$$

Another Example: Consider

$$f(x) = x^2 - 4x + 5$$

Let's try to compute $f'(3)$.

Idea: Use a second point nearby

Slope from 3 to 3.1	$\dfrac{f(3+0.1)-f(3)}{3.1-3} = \dfrac{[(3.1)^2-4(3.1)+5]-[(3)^2-4(3)+5]}{3.1-3} = \dfrac{2.21-2}{0.1} = \dfrac{0.21}{0.1} = 2.1$
Slope from 3 to 3.01	$\dfrac{f(3+0.01)-f(3)}{3.01-3} = \dfrac{[(3.01)^2-4(3.01)+5]-[(3)^2-4(3)+5]}{3.01-3} = \dfrac{2.0201-2}{0.01} = \dfrac{0.0201}{0.01} = 2.01$
Slope from 3 to 3.001	$\dfrac{f(3+0.001)-f(3)}{3.001-3} = \cdots = 2.001$
Slope from 3 to 3.0001	$\dfrac{f(3+0.0001)-f(3)}{3.0001-3} = \cdots = 2.0001$

It appears the secant slope is getting closer and closer to 2 as the second point gets closer. Now let's do this systematically with algebra:

First, shortcut: Instead of adding 0.1 or 0.01 or 0.001, let's just label this amount by a symbol: h

In each approximation, we were computing

$$f'(3) \approx \frac{f(3+h)-f(3)}{(3+h)-3} = ??$$

It becomes very easy to see the final answer if we can expand and simplify with algebra. Let's try it:

Recall: $f(x) = x^2 - 4x + 5$

What is $f'(3)$?

$\boxed{f'(3) = 2}$

Expand and completely simplify

$$\frac{f(3+h) - f(3)}{(3+h) - 3}$$

$$= \frac{\left[(3+h)^2 - 4(3+h) + 5\right] - \left[\overbrace{(3)^2 - 4(3) + 5}^{9 \;-\; 12 \;+5 \;=\; 2}\right]}{(3+h) \quad - \quad 3}$$

$$= \frac{(3+h)^2 - 12 - 4h + 5 \;-\; 2}{h}$$

$$(3+h)^2 = (3+h)(3+h)$$

$$= 9 + 3h + 3h + h^2$$

$$= 9 + 6h + h^2$$

$$= \frac{\cancel{9} + 6h + h^2 - \cancel{12} - 4h + \cancel{3}}{h}$$

$$= \frac{6h + h^2 - 4h}{h}$$

$$= \frac{2h + h^2}{h} \quad = \quad \frac{h(2+h)}{h} \quad = \quad 2 + h$$

$$= \frac{2h}{h} + \frac{h^2}{h}$$

$$= 2 + h$$

THUS, $\boxed{\dfrac{f(3+h) - f(3)}{h} = 2 + h}$

COMPARE WITH TABLE!

AS $h \to 0$, THIS SLOPE $\to 2$

Same function: $f(x) = x^2 - 4x + 5$ What is $f'(5)$?

Find $f'(5)$ by using the same process.

Expand and completely simplify

$$\frac{f(5+h) - f(5)}{(5+h) - 5}$$

$$= \frac{[(5+h)^2 - 4(5+h) + 5] - \overbrace{[(5)^2 - 4(5) + 5]}^{25 - 20 + 5 = 10}}{(5+h) - 5}$$

$$= \frac{\cancel{25} + 10h + h^2 - \cancel{20} - 4h + \cancel{5} - \cancel{10}}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= 6 + h$$

So $\boxed{f'(5) = 6}$

It is better just to do this once!

What is $f'(a)$?

Same function: $f(x) = x^2 - 4x + 5$

Find $f'(a)$ by using the same process.

Check: Does this match f'(3) and f'(5)?

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{[(a+h)^2 - 4(a+h) + 5] - [a^2 - 4a + 5]}{a + h \quad - a}$$

$$= \frac{\cancel{a^2} + 2ah + h^2 \cancel{- 4a} - 4h \cancel{+ 5} \cancel{- a^2} \cancel{+ 4a} \cancel{- 5}}{h}$$

$$= \frac{2ah + h^2 - 4h}{h}$$

$$= \frac{2ah}{h} + \frac{h^2}{h} - \frac{4h}{h}$$

$$= 2a + h - 4$$

So $\boxed{f'(a) = 2a - 4}$

$f'(3) = 2(3) - 4 = 2$ ✓

$f'(5) = 2(5) - 4 = 6$ ✓

YES!